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LAGRANGIAN SPHERES IN THE 2-DIMENSIONAL COMPLEX SPACE FORMS

BY

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ABSTRACT

By constructing a holomorphic cubic form for Lagrangian surfaces with nonzero constant length mean curvature vector in a 2-dimensional complex space form $\tilde{M}(4c)$, we characterize the Lagrangian pesudosphere as the only branched Lagrangian immersion of a sphere in $\tilde{M}(4c)$ with nonzero constant length mean curvature vector. When $c = 0$, our result reduces to Castro–Urbano's result in [1].

1. Introduction

An immersion $\phi : M \to N$ from an n-dimensional submanifold M to a $2n$-dimensional symplectic manifold (N, ω) is said to be **Lagrangian** if $\phi^*\omega = 0$, where ω is the symplectic form of N. When (N, ω) carries a Kähler structure, i.e., it possesses an integrable almost complex structure J such that the linear form

$$g(X, Y) := \omega(X, JY),$$

defines a Riemannian metric, the Lagrangian condition is equivalent to

$$J(\phi_* TM) \perp \phi_* TM.$$

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A known result by Dazord says that if $\mathbf{H}$ denotes the mean curvature vector field of a Lagrangian immersion ϕ to an Einstein–Kähler manifold, then the tangent vector field $J\mathbf{H}$ is a closed vector field on M [7]. This means that its dual 1-form $\alpha = \mathbf{H} \lrcorner \omega$, called the Maslov form of ϕ, is a closed form. Therefore, if M is a compact manifold and $H^1(M, \mathbb{R}) = 0$, there exists a smooth function f on M such that $df = \alpha$. Consequently, α, and so $\mathbf{H}$, vanish on at least two points. In particular, there are no Lagrangian (regular) immersions of two-spheres into an Einstein–Kähler manifold with mean curvature vector of non-null constant length.

In [1], Castro and Urbano studied branched Lagrangian immersions from two-spheres into $\mathbb{C}^2$. In fact, they obtained the following interesting result.

THEOREM 1.1 ([1]): *Let $\phi : M \to \mathbb{C}^2$ be a branched Lagrangian immersion of a sphere M. If the mean curvature vector $\mathbf{H}$ of ϕ has constant length, then $\phi(M)$ is congruent, up to dilatation, to the Lagrangian pseudosphere.*

It is natural to investigate the same problem in the case of non-flat complex space forms. The main results of this paper are in the following.

THEOREM 1.2: *Let $\phi : M \to \mathbb{CP}^2(4)$ be a branched Lagrangian immersion of a two sphere M. If the mean curvature vector $\mathbf{H}$ has nonzero constant length, then $\phi(M)$ is congruent, up to isometries, to the Lagrangian pseudosphere $\phi_1 : M \to \mathbb{CP}^2(4)$, which is given by Example 1.*

Remark 1: If ϕ is a minimal Lagrangian immersion from a two sphere in $\mathbb{CP}^2$, then by Yau's theorem in [9] we know that ϕ must be totally geodesic.

THEOREM 1.3: *Let $\phi : M \to \mathbb{CH}^2(-4)$ be a branched Lagrangian immersion of a two sphere M. If the mean curvature vector $\mathbf{H}$ has constant length, then $\phi(M)$ is congruent, up to isometries, to the Lagrangian pseudosphere $\phi_2 : M \to \mathbb{CH}^2(-4)$, which is given by Example 2.*

Combined with Castro and Urbano's result, our theorems can be interpreted in the spirit of the classical Hopf's theorem, characterizing the totally umbilical $(II - HI = 0)$ sphere as the only genus zero oriented surface with constant mean curvature in a 3-dimensional space form [6].

It is proved in [5] that there exist no totally umbilical Lagrangian submanifolds in a complex form $\tilde{M}^n(4c)$ with $n \geq 2$ except the totally geodesic ones. In view of this fact, Chen introduced the concept of **Lagrangian H-umbilical**

submanifolds as the "simplest" Lagrangian submanifolds next to the totally geodesic ones in complex space forms [3]. Instead of totally umbilical submanifolds, our Hopf-type theorems characterize the Lagrangian H-umbilical spheres ($\lambda = 2\mu$) as the only genus zero oriented surface with constant length mean curvature vector in 2-dimensional complex space forms.

2. Preliminaries

2.1. Lagrangian submanifolds and Legendrian submanifolds. If $\phi : M \to \mathbb{CP}^n$ (resp., $\mathbb{CH}^n$) is a Lagrangian immersion of a simply connected manifold M, then ϕ has a horizontal lift with respect to the Hopf fibration to $\mathbb{S}^{2n+1}$ (resp., $\mathbb{H}_1^{2n+1}$), which is unique up to isometries. We will denote this horizontal lift by $\tilde{\phi}$. Horizontal immersions from an n-dimensional manifold in $\mathbb{S}^{2n+1}$ (resp., $\mathbb{H}_1^{2n+1}$) are called Lengendrian immersions. It is known that Lagrangian immersions in $\mathbb{CP}^n$ (resp., $\mathbb{CH}^n$) are locally projections of Legendrian immersions in $\mathbb{S}^{2n+1}$ (resp., $\mathbb{H}_1^{2n+1}$).

2.2. Lagrangian H-umbilical submanifolds. An n-dimensional non-totally geodesic Lagrangian submanifold in a Kähler manifold is called a **Lagrangian H-umbilical submanifold** if its second fundamental form satisfies the following simple form:

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_2, e_2) &= \cdots = h(e_n, e_n) = \mu J e_1, \\ h(e_1, e_j) &= \mu J e_j, & h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \ldots, n, \end{aligned} \tag{2.1}$$

for suitable functions λ and μ with respect to some suitable orthonormal local frame fields $e_1, \ldots, e_n$. Such submanifolds can be regarded as the simplest Lagrangian submanifolds in a complex space form next to the totally geodesic ones.

Lagrangian H-umbilical submanifolds in complex Euclidean spaces satisfying (2.1) with $\lambda = 2\mu$ are determined in [2] as follows.

Theorem 2.1 ([2]): *Up to rigid motions of $\mathbb{C}^n$, a Lagrangian isometric immersion $\phi : M \to \mathbb{C}^n$ is a Lagrangian pseudosphere if and only if it is a Lagrangian H-umbilical immersion satisfying* (2.1) *with $\lambda = 2\mu$.*

Lagrangian H-umbilical submanifolds satisfying (2.1) with $\lambda = 2\mu$ in non-flat complex space forms have also been completely classified in [3] (see Theorems 5.1 and 6.1 in [3]). For simplification, we only present the results for $n = 2$.

THEOREM 2.2 ([3]): *Let $\phi : M \to \mathbb{CP}^2(4c)$ be a Lagrangian H-umbilical isometric immersion satisfying (2.1) with $\lambda = 2\mu$ for some nontrivial function μ, where $c > 0$, then*

(i) *μ is a constant,*
(ii) *M is an open portion of $\mathbb{S}^2(\delta^2)$ with $\delta^2 = \mu^2 + c$ and hence M is locally isometric to the warped product $I \times_{\frac{1}{\delta}\cos(\delta x)} \mathbb{S}^1(1)$,*
(iii) *up to rigid motions of $\mathbb{CP}^2(4c)$, the immersion ϕ is the composition $\pi \circ \tilde{\phi}$, where π is the projection of Hopf fibration from $\mathbb{S}^5(c)$ onto $\mathbb{CP}^2(4c)$ and $\tilde{\phi} : \hat{M} \to \mathbb{S}^5(c) \subset \mathbb{C}^3$ is given by*

$$\tilde{\phi}(x, y) = e^{i(\mu-\delta)x} z(y) + e^{i(\mu+\delta)x} w(y),$$

where

$$z(y) = \frac{1}{2\delta^2}\Big(\frac{\mu(\mu+\delta)}{\sqrt{c}} + \sqrt{c}\cos y, -(\mu+\delta) + \mu\cos y, \delta\sin y\Big),$$
$$w(y) = \frac{1}{2\delta^2}\Big(\frac{\mu(\mu-\delta)}{\sqrt{c}} + \sqrt{c}\cos y, -(\mu-\delta) + \mu\cos y, \delta\sin y\Big),$$

and $z : I \to \mathbb{S}^5\Big(\sqrt{\frac{2\delta c}{\delta+\mu}}\Big) \subset \mathbb{C}^3$ is a special Legendre curve with speed $1/(2\delta)$, $w : I \to \mathbb{S}^5\Big(\sqrt{\frac{2\delta c}{\delta-\mu}}\Big)$ is the associated special Legendre curve of z with speed $1/(2\delta)$ and $\hat{M}$ is the covering space of M via the Hopf fibration.

THEOREM 2.3 ([3]): *Let $\phi : M \to \mathbb{CH}^2(4c)$ be a Lagrangian H-umbilical isometric immersion satisfying (2.1) with $\lambda = 2\mu$ for some non-trivial function μ, where $c < 0$, then*

(i) *μ is a constant,*
(ii) *M is a real space form $M^2(K)$ of constant sectional curvature $K = \mu^2 + c$,*
(iii) *M is locally isometric to one of the following warped products:*

$$I \times_{1/\delta\cos(\delta x)} \mathbb{R}, \quad \mathbb{R} \times_1 \mathbb{R}, \quad \mathbb{R} \times_{e^{\delta x}} \mathbb{R},$$

and up to rigid motions of $\mathbb{CH}^2(4c)$, ϕ is the composition $\pi \circ \tilde{\phi}$, where π is the projection from $\mathbb{H}^5_1(c)$ onto $\mathbb{CH}^2(4c)$, $\delta = \sqrt{|K|}$ and (iii-1) *when* $K=\mu^2+c>0$, $\tilde\phi: M\to \mathbb{H}^5_1(c)\subset\mathbb{C}^3_1$ *is given by*

$$\tilde\phi(x,y)=\frac{e^{i\mu x}}{2\delta^2}\Big(\Big(\frac{\mu(\mu-\delta)}{\sqrt{-c}}-\sqrt{-c}\cos y\Big)e^{i\delta x}+\Big(\frac{\mu(\mu+\delta)}{\sqrt{-c}}-\sqrt{-c}\cos y\Big)e^{-i\delta x},$$

$$(\delta-\mu+\mu\cos y)e^{i\delta x}-(\delta+\mu-\mu\cos y)e^{-i\delta x},$$

$$\delta\sin y(e^{i\delta x}+e^{-i\delta x})\Big),$$

$$\delta=\sqrt{K};$$

(iii-2) *when* $K=\mu^2+c=0$, $\tilde\phi: M\to \mathbb{H}^5_1(c)\subset\mathbb{C}^3_1$ *is given by*

$$\tilde\phi(x,y)=e^{i\sqrt{-c}x}(1/\sqrt{-c}-ix+(\sqrt{-c}/2)y^2,x+(i/2)\sqrt{-c}y^2,y);$$

(iii-3) *when* $K=\mu^2+c<0$, $\tilde\phi: M\to \mathbb{H}^5_1(c)\subset\mathbb{C}^3_1$ *is given by*

$$\tilde\phi(x,y)=\frac{e^{i\mu x}}{2}\big((1/\sqrt{-c})(e^{\delta x}(1-(\mu/\delta)i-cy^2)+e^{-\delta x}(1+(\mu/\delta)i)),$$

$$e^{\delta x}((1/\delta)+(\mu i-\delta)y^2)-(1/\delta)e^{-\delta x},2ye^{\delta x}\big),$$

$$\delta=\sqrt{-K}.$$

Before we give proofs of Theorem 1.2 and Theorem 1.3, we first state two examples, which appeared in Theorem 2.2 for $c=1$ and Theorem 2.3 (iii-1) for $c=-1$.

Example 1 (Lagrangian pseudosphere $\phi_1: M\to\mathbb{CP}^2(4)$): ϕ_1 is given by the composition $\pi\circ\tilde\phi_1$, where π is the projection of Hopf fibration and $\tilde\phi_1: M\to\mathbb{S}^5(1)\subset\mathbb{C}^3$ is given by

$$\tilde\phi(x,y)=e^{i(\mu-\delta)x}z(y)+e^{i(\mu+\delta)x}w(y),$$

where

$$z(y)=\frac{1}{2\delta^2}(\mu(\mu+\delta)+\cos y,-(\mu+\delta)+\mu\cos y,\delta\sin y),$$
$$w(y)=\frac{1}{2\delta^2}(\mu(\mu-\delta)+\cos y,-(\mu-\delta)+\mu\cos y,\delta\sin y),$$

with

$$\mu=(2/3)|\mathbf{H}|,\quad \delta=\sqrt{1+\mu^2}.$$

Since $\det(g_{ij})=(1/\delta^2)\cos^2(\delta x)$, the set of singularities of ϕ is

$$\{(x,y)\in I\times\mathbb{R}|2\delta x=(2k+1)\pi,k\in\mathbb{Z}\}.$$

As ϕ is 2π-periodic in y, ϕ defines an immersion $\mathbb{R}\times\mathbb{S}^1$ in $\mathbb{CP}^2$. We restrict ϕ to the set

$$\{(x,y)\in I\times\mathbb{S}^1| -\pi\le 2\delta x\le\pi\},$$

so the singular points are isolated. Moreover,

$$\tilde{\phi}_1(-\pi/(2\delta),y)=-e^{-(ib\pi)/\delta}\tilde{\phi}_1(\pi/(2\delta),y),$$

hence $\phi_1(-\pi/(2\delta),y)=\phi_1(\pi/(2\delta),y)$. Using ϕ_1 is π/δ-periodic in x, if in $[-\pi/(2\delta),\pi/(2\delta)]\times\mathbb{S}^1$ we identify $\{-\pi/(2\delta)\}\times\mathbb{S}^1$ and $\{\pi/(2\delta)\}\times\mathbb{S}^1$ to two different points, we obtain a branched immersion from a two sphere to $\mathbb{CP}^2(4)$. We call $\phi_1: M\to\mathbb{CP}^2(4)$ the Lagrangian pseudosphere in $\mathbb{CP}^2$.

Example 2 (Lagrangian pseudosphere $\phi_2: M\to\mathbb{CH}^2(4)$): ϕ_2 is given by the composition $\pi\circ\tilde{\phi}_2$, where π is the projection of Hopf fibration from $\mathbb{H}^5_1(-1)$ onto $\mathbb{CH}^2(-4)$ and $\tilde{\phi}_2: M\to\mathbb{H}^5_1(-1)\subset\mathbb{C}^3_1$ is given by

$$\tilde{\phi}(x,y)=e^{i(\mu-\delta)x}z(y)+e^{i(\mu+\delta)x}w(y),$$

where

$$z(y)=\frac{1}{2\delta^2}(\mu(\mu+\delta)-\cos y,-(\mu+\delta)+\mu\cos y,\delta\sin y),$$
$$w(y)=\frac{1}{2\delta^2}(\mu(\mu-\delta)-\cos y,-(\mu-\delta)+\mu\cos y,\delta\sin y),$$

with

$$\mu=(2/3)|\mathbf{H}|,\quad \delta=\sqrt{\mu^2-1}.$$

By similar analysis, we obtain a branched immersion from a two sphere to $\mathbb{CH}^2(-4)$. We call $\phi_2: M\to\mathbb{CH}^2(-4)$ the Lagrangian pseudosphere in $\mathbb{CH}^2$.

3. Holomorphic cubic form

We consider now that the target manifold N is a simply connected 2-dimensional complex space form with complex structure J and constant holomorphic sectional curvature $4c$. We denote by $N=\tilde{M}(4c)$ the complex projective plane $\mathbb{CP}^2$ if $c=1$, the complex Euclidean plane $\mathbb{C}^2$ if $c=0$ and the complex hyperbolic plane $\mathbb{CH}^2$ if $c=-1$, with their standard complex structures J and metrics g.

Relative to the orthonormal frames

$$e_1,\, e_2;\, e_{1^*}=Je_1,\, e_{2^*}=Je_2,$$

the structure equations are

$$d\theta_A = \sum_B \theta_{AB} \wedge \theta_B, \quad \theta_{AB} + \theta_{BA} = 0, \tag{3.1}$$

$$d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{BC} + \Omega_{AB},$$

where the indices have the range

$$A, B, C, D, \ldots = 1, 2, 1^*, 2^*; \quad i, j, k, l, \ldots = 1, 2$$

and θ_A is an orthonormal coframe, $\theta_{AB}(= -\theta_{BA})$ are the connection forms, and

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\theta_C \wedge \theta_D,$$

where

$$\begin{aligned} K_{ABCD} =& c\{(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + g(Je_C, e_A)g(Je_D, e_B) \\ & - g(Je_C, e_B)g(Je_D, e_A) + 2g(Je_C, e_D)g(Je_B, e_A)\} \end{aligned} \tag{3.2}$$

are the curvature forms. Equation (3.2) expresses the fact that N is of constant holomorphic curvature $4c$. Denote by $\overline{\nabla}$ the connection of N with respect to g. It follows from the fact that $\overline{\nabla} \circ J = J \circ \overline{\nabla}$ that

$$\theta_{ij} = \theta_{i^*j^*},\ \theta_{i^*j} = \theta_{j^*i}. \tag{3.3}$$

Let $\phi : M \to \tilde{M}(4c)$ be a branched Lagrangian immersion from an oriented surface M into $\tilde{M}(4c)$. Outside of the branch points, we restrict to the frame $e_1, e_2; e_{1^*} = Je_1, e_{2^*} = Je_2$, such that e_1, e_2 are tangent to L. Then

$$\theta_{i^*} = 0,$$

and by (3.1),

$$\theta_{ji^*} = h^{i^*}_{jk}\theta_k, \tag{3.4}$$

where

$$h^{i^*}_{jk} = h^{i^*}_{kj}.$$

The first and second fundamental forms are respectively

$$I = \theta_1^2 + \theta_2^2,$$

$$II = \sum_{i=1}^{2}(h^{i^*}_{11}\theta_1^2 + 2h^{i^*}_{12}\theta_1\theta_2 + h^{i^*}_{22}\theta_2^2)e_{i^*}.$$

Taking the second formula of (3.3), we have

$$h_{jk}^{i*} = h_{ki}^{j*} = h_{ij}^{k*}. \tag{3.5}$$

The mean curvature vector is defined by

$$\mathbf{H} = \frac{1}{2}\sum_k H^{k^*} e_{k^*} = \frac{1}{2}\sum_k \Big(\sum_j h_{jj}^{k^*}\Big) e_{k^*}.$$

Exterior differentiation of (3.4) and use of (3.1) give

$$\sum Dh_{jk}^{i^*} \wedge \theta_k = 0,$$

where

$$Dh_{jk}^{i^*} = dh_{jk}^{i^*} + \sum_l h_{lk}^{i^*}\theta_{lj} + \sum_l h_{jl}^{i^*}\theta_{lk} + \sum_l h_{jk}^{l^*}\theta_{l^*i^*}.$$

By putting

$$Dh_{jk}^{i^*} = \sum h_{jk,l}^{i^*}\theta_l, \tag{3.6}$$

we get

$$h_{jk,l}^{i^*} = h_{jl,k}^{i^*}. \tag{3.7}$$

Thus $h_{jk,l}^{i^*}$ is symmetric in any two of its indices and

$$H_j^{i^*} = H_i^{j^*}, \tag{3.8}$$

where $H_j^{i^*}$ is defined by

$$\sum_j H_j^{i^*}\theta_j = dH^{i^*} + \sum_j H^{j^*}\theta_{ji}. \tag{3.9}$$

Choose an isothermal parameter $z = x + iy$ on M, denote the induced metric of M by $g = \rho^2 dz d\bar{z}$, so $\theta_1 = \rho dx, \theta_2 = \rho dy$. Write

$$\zeta = \theta_1 + i\theta_2 = \rho dz. \tag{3.10}$$

By (3.1) its exterior derivative is given by

$$d\zeta = i\zeta \wedge \theta_{12}. \tag{3.11}$$

Now suppose that the mean curvature vector $\mathbf{H}$ is nonzero everywhere. Define a cubic form on M by

$$\begin{aligned}\Theta =& 8\Big(\langle h(\phi_z,\phi_z), J\phi_z\rangle + \frac{2}{3|\mathbf{H}|^2}\langle \mathbf{H}, J\phi_z\rangle^3\Big) \otimes (dz)^3\\ =& \hat{H}\zeta^3,\end{aligned}$$

where

$$\hat{H} = (h_{11}^{1^*} - 3h_{22}^{1^*}) + i(h_{22}^{2^*} - 3h_{11}^{2^*}) + \frac{1}{12|\mathbf{H}|^{\mathbf{2}}}(H^{1^*} - iH^{2^*})^3 \tag{3.12}$$

and $\langle \, , \rangle$ denotes the inner product associated with the Riemannian metrics on M as well as on $\tilde{M}(4c)$.

Remark 2: It is easy to see that Θ is independent of the choice of the complex coordinates. So it is a global cubic form on M. It is a generalization of the cubic form for Lagrangian surface in $\mathbb{C}^2$ given by Castro and Urbano in [1].

THEOREM 3.1: *If H is a nonzero constant, Θ is a holomorphic cubic form on M.*

Proof. The hypothesis implies

$$H^{1^*}H^{1^*}_{,k} + H^{2^*}H^{2^*}_{,k} = 0, \quad k = 1, 2. \tag{3.13}$$

From (3.13), (3.5), (3.6), (3.7), (3.8) and (3.9), it follows that

$$\begin{aligned} d\hat{H} =& 3\theta_{12}\big\{3h_{11}^{2^*} - h_{22}^{2^*} + \frac{1}{4|\mathbf{H}|^2}(H^{1^*})^2H^{2^*} - \frac{1}{12|\mathbf{H}|^2}(H^{2^*})^3\big\} \\ &+ 3i\theta_{12}\big\{-3h_{22}^{1^*} + h_{11}^{1^*} + \frac{1}{12|\mathbf{H}|^{\mathbf{2}}}(H^{1^*})^3 - \frac{1}{4|\mathbf{H}|^2}H^{1^*}(H^{2^*})^2\big\} \\ &- \zeta\big\{4h_{221}^{1^*} + ih_{111}^{2^*} - ih_{221}^{2^*} + \frac{1}{2|\mathbf{H}|^2}H^{1^*}H^{2^*}H_2^{1^*} \\ &\quad + i\frac{1}{4|\mathbf{H}|^2}[(H^{1^*})^2 - (H^{2^*})^2]H_2^{1^*}\big\} \end{aligned} \tag{3.14}$$

From (3.10) and the definition of Θ, we have

$$\Theta = \hat{H}\rho^3(dz)^3.$$

It suffices to show that the coefficient $\hat{H}\rho^3$ of $(dz)^3$ in this expression is a holomorphic function of z. By substituting (3.10) into (3.11), we get

$$d\rho + i\rho\theta_{12} \equiv 0 \mod dz,$$

while (3.14) implies

$$d\hat{H} - 3i\theta_{12}\hat{H} \equiv 0 \mod dz.$$

It follows that

$$d(\hat{H}\rho^3) \equiv 0 \mod dz.$$

Therefore, for a branched Lagrangian immersion $\phi : M \to \tilde{M}(4c)$ with the nonzero constant length mean curvature vector, the cubic form Θ is holomorphic outside of the branch points. Moreover, from the expression of Θ, we know that Θ has zeros at the branch points. So Θ has no poles and it is a holomorphic 3-differential on M. ∎

Remark 3: It is well-known that if $\phi : M \to \tilde{M}(4c)$ is a minimal surface in a 2-dimensional complex space form, i.e., $\mathbf{H} \equiv 0$, then

$$\Theta = 8\langle h(\phi_z, \phi_z), J\phi_z\rangle (dz)^3$$

is a holomorphic 3-form on M.

PROPOSITION 3.2: *Let $\phi : M \to \tilde{M}(4c)$ be a branched Lagrangian immersion from an oriented surface M into a 2-dimensional complex space form $\tilde{M}(4c)$. Then $\Theta \equiv 0$ if and only if M is a Lagrangian H-umbilical surface satisfying*

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_2, e_2) = \mu J e_1, \quad h(e_1, e_2) = \mu J e_2,$$

with $\lambda = 2\mu$ with respect to some suitable orthonormal local frame field.

Proof. It is clear that $\lambda = 2\mu$ implies $\Theta \equiv 0$. Conversely, it follows from (3.12) that $\Theta \equiv 0$ is equivalent to

$$\begin{aligned} 12|\mathbf{H}|^2(h_{11}^{1^*} - 3h_{22}^{1^*}) + (H^{1^*})^3 - 3H^{1^*}(H^{2^*})^2 = 0, \\ 12|\mathbf{H}|^2(h_{22}^{2^*} - 3h_{11}^{2^*}) + (H^{2^*})^3 - 3(H^{1^*})^2 H^{2^*} = 0. \end{aligned}$$

It can be deduced to

$$\begin{aligned} 12|\mathbf{H}|^2 h_{11}^{1^*} &= 2(H^{1^*})^3 + 3H^{1^*}(H^{2^*})^2, \\ 12|\mathbf{H}|^2 h_{22}^{1^*} &= (H^{1^*})^3, \\ 12|\mathbf{H}|^2 h_{11}^{2^*} &= (H^{2^*})^3, \\ 12|\mathbf{H}|^2 h_{22}^{2^*} &= 3(H^{1^*})^2 H^{2^*} + 2(H^{2^*})^3. \end{aligned}$$

Take the local orthonormal frame $e_1, e_2, e_{1^*}, e_{2^*}$ such that $e_{1^*} = \mathbf{H}/|\mathbf{H}|$, then $H^{1^*} = 2|\mathbf{H}|$ and $H^{2^*} = 0$. It is easy to get that

$$h(e_1, e_1) = \lambda e_{1^*}, \quad h(e_2, e_2) = \mu e_{1^*}, \quad h(e_1, e_2) = \mu e_{2^*},$$

where $\lambda = (4/3)|\mathbf{H}| = 2\mu$. ∎

As a corollary, we have

THEOREM 3.3: *Let $\phi : \mathbb{S}^2 \to \tilde{M}(4c)$ be a branched Lagrangian immersion from a two sphere to a 2-dimensional complex space form with constant length mean curvature vector. Then the surface is Lagrangian H-umbilical with $\lambda = 2\mu$.*

Taking account of Theorem 2.1, we get the following corollary, which is the main result in [1].

COROLLARY 3.4: *Let $\phi : M \to \mathbb{C}^2$ be a branched Lagrangian immersion from a two sphere with constant length mean curvature vector. Then M is a Lagrangian pseudosphere.*

4. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. Let $\phi : M \to \mathbb{CP}^2(4)$ be a branched Lagrangian immersion from a two sphere with constant length mean curvature vector. Then it follows from Theorem 2.2 and Theorem 3.3 that up to isometries, M is the Lagrangian pseudosphere $\phi_1 : M \to \mathbb{CP}^2(4)$, which is given by Example 1. Therefore we complete the proof of Theorem 1.2. ∎

We also need the following lemma in order to prove our Theorem 1.3.

LEMMA 4.1 ([8]): *Let $\phi : \Sigma \to N \subset \mathbb{R}^k$ be a conformal smoothly branched immersion from a compact Riemann surface Σ. Then*

$$\frac{1}{2\pi}\int_\Sigma K dv = \chi(\Sigma) + b,$$

where $\chi(\Sigma)$ is the Euler number of Σ and b the number of branch points of Σ, counted with multiplicities.

Proof of Theorem 1.3. Let $\phi : M \to \mathbb{CH}^2(-4)$ be a branched Lagrangian immersion from a two sphere with constant length mean curvature vector. According to the above generalized Gauss-Bonnet formula, there is no branched immersion from a topological two sphere to $\mathbb{CH}^2(-4)$ with non-positive curvature. Then it follows from Theorem 2.3 and Theorem 3.3 that up to isometries, M is the Lagrangian pseudosphere $\phi_2 : M \to \mathbb{CH}^2(-4)$, which is given by Example 2. This completes the proof of Theorem 1.3. ∎

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